Lecture 19

Wednesday, 9 November 2022 11:37 AM

Revenue - Maximizing Auctions.

Suppose our goal is to maximize the revenue, or the payment, we get in an auction.

This is somewhat different from the mech. design we have seen until now. There is no SCF that maximizes payments. So we have to step away from mech. design, & focus on auctions. But many of the tools will still be useful.

Consider for instance the simple case of one item, one bidder. For maximizing SW, this is trivial. But what if we want to maximize revenue? Again, clearly if we have no information about the bidder's value, we cannot possibly get maximum revenue (or even non-zero revenue) in every instance. Hence instead, we will focus on maximizing revenue "on average."

Bayesian Single Parameter Setting:

- n agents / bidders
- bidder i's PRIVATE value v_i is drawn from a PUBLIC distribution with cdf F_i
- bidder i's bids are chosen from $supp(F_i)$
- Feasible allocation set $A \subseteq \mathbb{R}^n$ (for $x \in A$, x_i is amt. of item that bidder i gets)
- Design mechanism $M = (x, p)$

  bidder i's utility $u_i(b_i, b_{-i}) = v_i x_i(b) - p_i(b)$

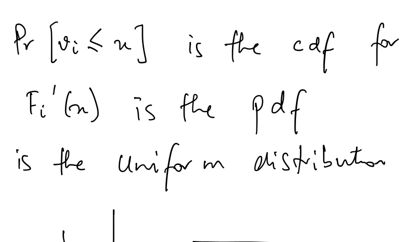
Note:

1. We still want DSIC mechanisms. Hence, Myerson's Lemma still holds: x must be monotone, & $p_i(b_i) = \int_{w \in supp(F_i)} w x_i'(w)\, dw$

2. $F_i(x) = Pr[v_i \leq x]$ is the cdf for value of bidder i

   $f_i(x) = F_i'(x)$ is the pdf

   $U[a,b]$ is the uniform distribution in $[a,b]$



$$f(x) = 1/(b-a) \quad \text{if } x \in [a,b]$$
$$= 0 \quad \text{o.w.}$$

$$F(x) = \min\left\{1, \max\left\{0, \frac{x-a}{b-a}\right\}\right\} \quad \text{for } x \in \mathbb{R}_+$$

3. $F = F_1 \times \dots \times F_n$ is the joint distribution for values

   $v \sim F$ is a sampled tuple of values, $v_i \sim F_i$

Example:

Consider again the single item, single bidder setting, $v \sim U[0,1]$

Consider the auction: $x(b) = 1$ if $b \geq 1/2$

$= 0$ o.w.

then $p(b) = 1/2$ if $b \geq 1/2$

$= 0$ o.w.

(from Myerson's Lemma)

& $\mathbb{E}_{b \sim U[0,1]}[p(b)] = 1/4$ is the expected revenue

- this is called a "posted price" auction: bidder takes if she wants, pays the posted price (1/2 in this case)

  also a "reserve price" auction: item only gets given if $v_i \geq$ reserve price (1/2 in this case)

Q. Can we do better? i.e., get more revenue in expectation, w/ a DSIC mechanism?

Example 2:

Consider single item, 2 bidders, $v_i \sim U[0,1]$ for both.

For VSP, revenue $= \min\{v_1, v_2\}$

let $X = \min\{v_1, v_2\}$. Then
$$F_X(x) = Pr[X \leq x]$$
$$= 1 - Pr[X > x]$$
$$= 1 - Pr[v_1 > x \;\&\; v_2 > x]$$
$$= 1 - (1-x)^2$$
$$= 2x - x^2$$

Hence $f_X(x) = 2 - 2x$
$$\mathbb{E}[X] = \int_{x=0}^{1} x \cdot f_X(x)\, dx = \int_0^1 (2x - 2x^2)\, dx$$
$$= \left| x^2 - \frac{2}{3}x^3 \right|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

However, now consider the following reserve price auction:
$$x_i(b_i) = \begin{cases} 1 & \text{if } b_i \geq \max\{\frac{1}{2}, b_{-i}\} \\ 0 & \text{o.w.} \end{cases}$$

Then $p_i(b_i) = \max\{\frac{1}{2}, b_{-i}\}$ if $x_i(b_i) = 1$

$= 0$ o.w.

(note that this is a reserve price mechanism, but not a posted-price mechanism)

The expected revenue of this auction is 5/12!
(Prove yourself)

Hence, somewhat counterintuitively, sometimes not allocating the item can lead to an increase in revenue.

Q. Can we do better than 5/12?

Myerson showed that for monotone allocation rules, there is a very nice characterization for the expected revenue.

Fix bidder i, b_{-i}. As before, we write $x_i(z), p_i(z)$
$= x_i(z, b_{-i}), p_i(z, b_{-i})$

Recall that for DSIC mechanisms,
$$p_i(b_i) = \int_{z=v_{min}}^{b_i} z\, x_i'(z)\, dz$$

Assuming DSIC mechanism, $b_i = v_i$, hence
$$\mathbb{E}_{v_i \sim F_i}[p_i(b_i)] = \mathbb{E}_{v_i \sim F_i}[p_i(v_i)]$$
$$= \int_{v_i = v_{min}}^{v_{max}} f_i(v_i)\, p_i(v_i)\, dv_i$$
$$= \int_{v_i = v_{min}}^{v_{max}} f_i(v_i) \int_{z=v_{min}}^{v_i} z\, x_i'(z)\, dz\, dv_i$$
$$= \int_{z=v_{min}}^{v_{max}} \int_{v_i = z}^{v_{max}} f_i(v_i)\, z\, x_i'(z)\, dz\, dv_i$$
$$= \int_{z=v_{min}}^{v_{max}} \underbrace{(1 - F_i(z))\, z}_{u}\, \underbrace{x_i'(z)}_{v}\, dz$$

Using IBP, $\int uv = u \int v - \int u' \int v$

$$\mathbb{E}[p_i(v_i)] = \left[(1 - F_i(z))\, z\, x_i(z) - \int \left((1 - F_i(z)) - z f_i(z) \right) x_i(z)\, dz \right]_{v_{min}}^{v_{max}}$$

Lets assume $x_i(v_{min}) = 0$

Then
$$\mathbb{E}[p_i(v_i)] = \int_{v_{min}}^{v_{max}} x_i(z) \left[z f_i(z) - (1 - F_i(z)) \right] dz$$
$$= \int_{v_{min}}^{v_{max}} x_i(z) f_i(z) \left[z - \frac{1 - F_i(z)}{f_i(z)} \right] dz$$

Define "virtual valuation" $\varphi_i(z) = z - \frac{1 - F_i(z)}{f_i(z)}$

Then
$$\mathbb{E}[p_i(v_i)] = \int_{v_{min}}^{v_{max}} \varphi_i(z)\, x_i(z)\, f_i(z)\, dz$$
$$= \mathbb{E}_{v_i \sim F_i}[\varphi_i(v_i)\, x_i(v_i)]$$

This for bidder i.
$$\mathbb{E}_{v \sim F}\left[\sum_i p_i(v_i)\right] = \sum_i \mathbb{E}_{v_{-i} \sim F_{-i}}\left[\mathbb{E}_{v_i \sim F_i}[p_i(v_i)]\right]$$
$$= \sum_i \mathbb{E}_{v \sim F}[\varphi_i(v_i)\, x_i(v_i)]$$
$$= \mathbb{E}_{v \sim F}\left[\sum_i \varphi_i(v_i)\, x_i(v_i)\right]$$

Thus, the expected revenue is exactly $\mathbb{E}_{v \sim F}\left[\sum_i \varphi_i(v_i)\, x_i(v_i)\right]$,

for any DSIC mechanism (w/ monotone x)

where $\varphi_i(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}$

Thus to maximize revenue, we must choose monotone x that maximizes $\mathbb{E}_{v \sim F}\left[\sum_i \varphi_i(v_i)\, x_i(v_i)\right]$, or the "expected virtual social welfare"

Example: if $v_i \sim U[0,1]$,
$$\varphi_i(v_i) = v_i - \frac{1 - v_i}{1} = 2v_i - 1, \quad \text{for } v_i \leq 1$$

Consider the mechanism:
$$x(b) = \arg\max_{y \in X} \sum_i \varphi_i(b_i)\, y_i(b)$$

i.e., maximizes the "virtual" social welfare.

Clearly, if x is monotone, this is optimal revenue-maximizing DSIC mechanism.

Q. is x monotone? If $\varphi_i(b_i)$ is monotone in b_i, then yes!

Note that this is a function of the distribution F_i.
E.g., uniform distributions, exponential distributions, Gaussian distributions are regular.
But multimodal distributions or heavy-tailed distributions are not.
Distributions for which $\varphi_i(v_i)$ is monotone, are called regular distributions (or Myerson regular distributions)

Example: Single bidder, $v \sim U[0,1]$

Maximizing $\varphi(v)\, x(v) = (2v - 1)\, x(v)$ gives us:

$x(v) = 1$ if $v \geq 1/2$

$= 0$ o.w.

Hence, our earlier auction is indeed optimal, among DSIC auctions

Example: single item, 2 bidders, $v_1 \sim U[0,1]$, $v_2 \sim U[0,2]$

then $\varphi_1(v_1) = 2v_1 - 1$, $\varphi_2(v_2) = 2v_2 - 2$

To maximize expected revenue, we choose x to maximize $x_1(2v_1 - 1) + x_2(2v_2 - 2)$

Hence, the optimal x is:
- give item to bidder 1 if $b_1 \geq \frac{1}{2}$, $b_1 \geq b_2 - 1/2$
- give item to bidder 2 if $b_2 \geq 1$, $b_2 \geq b_1 + 1/2$
- else allocate to neither.